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DETERMINATION OF THE SHAPE OF THE EARTH'S PHYSICAL SURFACE FROM THE ANOMALIES OF THE VERTICAL GRADIENT OF GRAVITATIONAL ACCELERATION

S. V. Gromov

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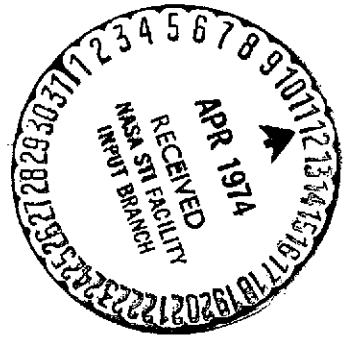

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16. Abstract

The anomaly of the vertical gradient of gravitational acceleration is used to determine the shape of the physical surface of the Earth. The boundary condition for the perturbing potential and the basic integral equations are derived. A calculation is made of the correction factor in the basic boundary condition.

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S. V. Gromov

/ 26*

The classical dynamic method of determining the shape of the geoid proposed by Stokes in 1849 [8], was based on the use of the values of the first (vertical) derivative of the potential W of gravitational acceleration at the points of the surface being determined. M. S. Molodebtskiy [4] solved the problem of determining the shape of the physical surface of the Earth. This solution is also based on the distribution of the anomalies of gravity at points of the surface being determined. Along with great theoretical value, it is extremely important in practical terms since at the present time of all the derivatives of the potential W = const, we can best measure the first vertical derivatives (g).

However, there is still interest in the methods for determining the shape of the Earth which use other derivatives of W, whose mass measurement is not reliable at the given time, or has insufficient accuracy. In 1949, N. K. Migal' [2] provided a rigorous solution for the problem of determining the shape of the level surface of the gravity potential by means of the sum of the main radii of curvature of the latter. The values for the main radii

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* Numbers in the margin indicate pagination of original foreign text.

of curvature for the surface W = const may be obtained at each point by means of the gravitational acceleration (g) and the derivatives $\frac{\partial^2 W}{\partial x \partial y}$, $\left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2}\right)$, as well as by the second vertical derivative of the gravity potential $\frac{\partial^2 W}{\partial z^2}$ or, as it is frequently called, the vertical gradient p of the force of gravity. In practice, this solution is of little value, since insurmountable difficulties are encountered when reducing the measured derivatives from the physical surface of the Earth to the level surface in the field of the real Earth. In addition, to obtain the necessary data it is necessary to make complex measurements at each point with different devices of at least 3 types: a gravimeter (g), a gravitational variometer $\left(\frac{\partial^2 W}{\partial y^2} - \frac{\partial^2 W}{\partial x^2}\right)$, and a vertical gradient meter $\left(\frac{\partial^2 W}{\partial z^2}\right)$. The last device is still in a stage of development. In 1966 I. F. Monin [6] solved the problem of determining the shape of the physical surface of the Earth S by means of the derivative of the mixed anomaly of gravity $(g - \gamma)$ in the direction of the normal to the level ellipsoid (normal earth). This derivative is not purely a physical quantity (γ is the gravitational acceleration in an imaginary gravitational field) and cannot be obtained by performing measurements with any equipment. In another article I. F. Monin showed that this derivative equals /27

$$\Delta p = \quad . \qquad (1.0)$$

where Δp is the mixed anomaly of the vertical gradient of gravity, ρ - radius vector of the corresponding point on the auxiliary surface. It is true that the Equation (1.0) for the derivative being discussed is insufficiently accurate, since its author did not make a distinction between the direction of the normal to a level ellipsoid and a perpendicular line. For points on the physical surface of the Earth, in individual cases, this difference may be substantial 1', and the error in (1.0) may be on the order

of 0.1E, which exceeds by several factors the required accuracy for the measurement of the vertical gravitational gradient. Therefore the solution given by I. F. Monin [6] requires, for even an approximate realization, a change in two different physical quantities at points on the surface of the Earth: the force of gravity and its vertical gradient. In this same work [6] I. F. Monin, using the Stokes-Molodebtskiy boundary condition, excluded the gravitational anomaly from (1.0). He obtained an approximate boundary condition, connecting the perturbing potential and Δp. The external boundary value problem of potential theory is solved with this approximate boundary condition. The solution is reduced to solving a series of integral equations of the first kind.

Somewhat before the studies of I. F. Monin, a work was published [1] in which a solution was given for the problem of determining the geoid shape by means of the anomaly Δp of the vertical gravitational gradient. We shall attempt to solve a similar problem with respect to determining the shape of the physical surface of the Earth. Since the boundary condition in the Stokes-Molodebtskiy solution is obtained in an approximate form, we shall not use it as the initial condition when obtaining the boundary condition in our problem. In our boundary condition, we shall omit these values, since it is necessary to differentiate the approximate expressions.

1. Boundary Condition for the Perturbing Potential

Let us assume the point A is close to the physical surface of the Earth S, outside of it (see the figure). Let us examine 3 directions at A: z — the direction of the outer normal to the surface $W = W_0$, passing through A; n — direction of the outer normal from A to the level ellipsoid of rotation E_0; the direction ξ of the outer normal to the level surface $U = U'_0$ passing through

A. The level ellipsoid E_o is oriented in the field of the Earth in an absolute manner. The angular rotational velocities of the Earth and the ellipsoid E_o are assumed to be equal, and the rotational axes are assumed to coincide. In this case the perturbing potential T=W-U where U- is the normal potential. The effect of rotation of the Earth and the ellipsoid E_o from T will be excluded, and T is a harmonic function outside of S and E_o and is regular at infinity. The dimensions and forms of the surface E_o are selected so that it is sufficiently close to the geoid surface. It is assumed that the point A is the origin of three systems of rectangular coordinates: m, q, n; x, y, z and ξ, η, ζ. The axes m, x, ξ are selected in the planes of the meridians of the locations (northerly direction) corresponding to the normals n, z, ζ; the axes q. y. η go in an easterly direction. /28

We can determine the position of any point in space in any of the three coordinate systems, and there are several relations between the coordinates, for example:

$$\begin{aligned} &\alpha_1 x + \beta_1 y + \gamma_1 z, \\ &\alpha_2 x + \beta_2 y + \gamma_2 z, \\ &\alpha_3 x + \beta_3 y + \gamma_3 z. \end{aligned} \qquad (1.1)$$

$$\begin{aligned} \xi &= a_1 x + b_1 y + c_1 z \\ \eta &= a_2 x + b_2 y + c_2 z, \\ \zeta &= a_3 x + b_3 y + c_3 z. \end{aligned} \qquad (1.2)$$

where $\alpha_i, \beta_i, \gamma_i, a_i, b_i, c_i$ are the direction cosines, i = 1,2,3. The potential W of the gravity of the real Earth may be written in the form $W(x, y, z)$

$$W[\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z)] = T[m(x, y, z), q(x, y, z), n(x, y, z)] \qquad (1.3)$$

Differentiating (1.3) with respect to z, and taking (1.1) and (1.2) into account, we obtain

$$\frac{\partial W}{\partial z} = \frac{\partial T}{\partial m}\gamma_1 + \frac{\partial T}{\partial q}\gamma_2 + \frac{\partial T}{\partial n}\gamma_3 + \frac{\partial U}{\partial \xi}c_1 + \frac{\partial U}{\partial \eta}c_2 + \frac{\partial U}{\partial \zeta}c_3 \qquad (1.4)$$

Differentiating (1.4) again with respect to z, we may write

$$\frac{\partial^2 W}{\partial z^2} = \gamma_1\left|\frac{\partial^3 T}{\partial m^3}\gamma_1 + \frac{\partial^3 T}{\partial m^2 \partial q}\gamma_2 + \frac{\partial^3 T}{\partial m \partial n}\gamma_3\right| + c_1\left|\frac{\partial^2 U}{\partial \xi^2}c_1 + \frac{\partial^2 U}{\partial \xi \partial \eta}c_2 + \frac{\partial^2 U}{\partial \xi \partial \zeta}c_3\right|$$
$$+ \gamma_2\left|\frac{\partial^3 T}{\partial q \partial m}\gamma_1 + \frac{\partial^3 T}{\partial q^2}\gamma_2 + \frac{\partial^3 T}{\partial q \partial n}\gamma_3\right| + c_2\left|\frac{\partial^2 U}{\partial \eta \partial \xi}c_1 + \frac{\partial^2 U}{\partial \eta^2}c_2 + \frac{\partial^2 U}{\partial \eta \partial \zeta}c_3\right| + \qquad (1.5)$$
$$+ \gamma_3\left|\frac{\partial^3 T}{\partial n \partial m}\gamma_1 + \frac{\partial^3 T}{\partial n \partial q}\gamma_2 + \frac{\partial^3 T}{\partial n^2}\gamma_3\right| + c_3\left|\frac{\partial^2 U}{\partial \zeta \partial \xi}c_1 + \frac{\partial^2 U}{\partial \zeta \partial \eta}c_2 + \frac{\partial^2 U}{\partial \zeta^2}c_3\right|.$$

The components in the right wide of (1.5) have differing values, some of which may be omitted in general. The angles θ, θ' between the axes z and ζ and also z and n at points on the surface of the Earth and immediately adjacent to it can reach values of 1'; the angle $\bar{\mu}$ between the axes ζ and n equals the correction for curvature of the force line in a normal field. It is in all 1.5" for maximum altitudes above the Earth. The second derivitives at A in (1.5) had the following values: $\frac{\partial^2 U}{\partial \xi \partial \eta} = \frac{\partial^2 U}{\partial \eta \partial \zeta} = 0$; $\frac{\partial^2 U}{\partial \xi^2} \cong 8L$, $\frac{\partial^2 U}{\partial \eta^2} \simeq -1500E$;

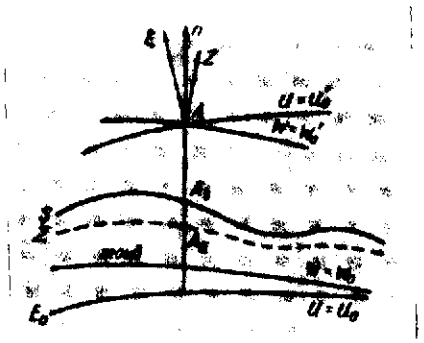
$$\frac{\partial^2 U}{\partial \eta^2} \simeq -1500E; \quad \frac{\partial^2 U}{\partial \zeta^2} \simeq 3000E \quad \text{(Eötvös)}$$

The derivatives $\frac{\partial^3 T}{\partial m^3}, \frac{\partial^3 T}{\partial q^3}, \frac{\partial^3 T}{\partial m^2 \partial q}, \frac{\partial^3 T}{\partial m \partial n^2}, \frac{\partial^3 T}{\partial q \partial n^2}$ can assume values reaching 100 - 200 and more Eötvös. Taking the fact into account that $\alpha_1 \simeq \beta_2 \simeq \gamma_3 \simeq a_1 \simeq b_2 \simeq c_3 \simeq 1 - \frac{\theta^2}{2}$, and the remaining sines are on the order of θ, we may write the following at the point A:

$$\frac{\partial^2 W}{\partial z^2} = 2\gamma_1 \frac{\partial^2 T}{\partial m \partial n} + 2\gamma_2 \frac{\partial^2 T}{\partial q \partial n} + \gamma_3^2 \frac{\partial^2 T}{\partial n^2} + \frac{\partial^2 U}{\partial \zeta^2}, \qquad (1.6)$$

and the error in this equation is less than 0.005 E. Completing the errors $\simeq 1{,}10^{-1} L$, we may write

$$\gamma_3^2 \frac{\partial^2 T}{\partial n^2} = 2\gamma_3 \frac{\partial^2 T}{\partial n^2} - \frac{\partial^2 T}{\partial n^2}.$$



Approximate distribution of certain surfaces and normals to them at the point A.

Assuming that $\gamma_1 \frac{\partial^2 T}{\partial m \partial n} + \gamma_2 \frac{\partial^2 T}{\partial q \partial n} + \gamma_3 \frac{\partial^2 T}{\partial n^2} = \frac{\partial^2 T}{\partial n \partial z}$, we may write

$$\frac{\partial^2 W}{\partial z^2} = \frac{\partial^2 U}{\partial z^2} + 2\frac{\partial^2 T}{\partial z \partial n} - \frac{\partial^2 T}{\partial n^2}. \qquad (1.7)$$

/29

After setting A at A_S, with the same accuracy we obtain

$$\frac{\partial^2 W}{\partial z^2} \qquad (1.8)$$

however,

$$\qquad (1.9)$$

where N is the anomaly of altitude or the altitude of the quasigeoid above the ellipsoid E_o with a neglibly small error, in (1.9) we can replace $\frac{\partial^2 T}{\partial n^2}$ by the normal vertical gravitational gradient $\frac{\partial \gamma}{\partial h}$. We can express N in terms of the Burns formula [3] and after substituting in (1.8) we obtain

$$\Delta p = \left(2\frac{\partial^2 T}{\partial z \partial n} - \frac{\partial^2 T}{\partial h^2}\right)_S + \frac{\partial^2 U}{\partial n^2}\left(\frac{T_S + U_o - W_o}{\gamma_E}\right), \qquad (1.10)$$

where $\Delta p = \frac{\partial^2 W}{\partial z^2} - \frac{\partial^2 U}{\partial z^2}$ is the mixed anomaly of the vertical gradient of gravitational acceleration, γ_E — normal value of

gravitational acceleration at the point A_Σ. Assuming the error in (1.10) is no more than 0.0005E, we can replace $\frac{\partial U}{\partial n}\Big|$ by $\frac{\quad}{\quad}$.

We assume that a sphere with a certain radius R is the reference surface E_o. In this case, instead of (1.10) we obtain the equation

$$\left|2\frac{\partial^2 T}{\partial z \partial \rho} - \frac{\partial^2 T}{\partial \rho^2}\right|_0 - \frac{6T_S}{\rho_\Sigma^2} = \Delta g + \frac{6(U_0 - W_0)}{\rho_\Sigma^2}, \qquad (1.11)$$

where ρ is the distance measured from the center of the reference sphere. Since the physical surface of the Earth is not known, in the first approximation we can assume that condition (1.11) pertains to an auxiliary surface Σ [3], obtained by plotting the normal altitudes from E_o. We thus find that it is necessary to have a solution of the outer boundary value problem of potential theory with the boundary condition

$$\frac{\partial^2 T}{\partial \rho^2} - \frac{6T}{\rho^2} + 2\gamma_1 \frac{\partial^2 T}{\partial n \partial \rho} + 2\gamma_2 \frac{\partial^2 T}{\partial q \partial \rho} = \Delta g + \frac{6(U_0 - W_0)}{\rho^2}. \qquad (1.12)$$

2. Basic Integral Equation

The direction cosines γ_1 and γ_2 are also projections of the angle between the axes n and z and the plane nAm and nAq. Finding the error, which equals the correction for curvature of the force line in a normal field (no more than 1.5"), we may write

$$\gamma_1 = -\frac{1}{\gamma}\frac{\partial T}{\partial m}, \quad \gamma_2 = -\frac{1}{\gamma}\frac{\partial T}{\partial q}.$$

After this, the boundary condition (2.12) assumes the form

$$\frac{\partial^2 T}{\partial \rho^2} - \frac{6T}{\rho^2} - \frac{2}{\gamma}\left|\frac{\partial T}{\partial m}\cdot\frac{\partial^2 T}{\partial m \partial \rho} + \frac{\partial T}{\partial q}\cdot\frac{\partial^2 T}{\partial q \partial \rho}\right| = -\delta_g, \qquad (2.1)$$

with $\delta_g = \Delta g + \frac{6(U_0 - W_0)}{\rho^2}$. The outer form of the boundary condition (2.1) indicates the difficulty of solving the problem of determining the potential T. It is true that the solution can be

simplified by the fact that the third term in the right side of (2.1) is small and barely reaches 0.2 - 0.3 E. In level regions, when there are no sharp density jumps in the mass of the Earth's core, it can be disregarded in general. Therefore, we have the following way to solve the problem: in the first approximation, we solve the boundary value problem, neglecting the third term in (2.1). Finding the potential T in the first approximation, we determine $\frac{\partial T}{\partial m}$, $\frac{\partial T}{\partial h}$, $\frac{\partial T}{\partial m \partial \varphi}$, $\frac{\partial^2 T}{\partial \varphi \partial \lambda}$, and substitute them in (2.1). After transferring these terms to the right side of (2.1), we again obtain a condition of the following form / 30

$$\frac{\partial^2 T}{\partial \rho^2} - \frac{6T}{\rho^2} = -\delta p. \qquad (2.2)$$

It must be assumed that it is not necessary to go beyond the second approximation. Actual calculations with observational data can give an accurate answer to this question. We must now solve the outer boundary value problem of potential theory with the simpler boundary condition (2.2). For this purpose, we represent the harmonic function outside of Σ, which is regular at infinity $\rho \frac{\partial T}{\partial h}$ [7], in the form of the potential of a simple layer of a certain density μ distributed on the surface of the Earth Σ of the first approximation, i.e., we may write

$$\rho \frac{\partial T}{\partial \rho} = \int_{\Sigma} \frac{\mu}{r} d\Sigma, \qquad (2.3)$$

where $r = \rho' = \sqrt{\rho^2 - 2\rho\rho'\cos\psi}$, ρ' — is the radius vector of a point on Σ, ρ — radius vector of a fixed point Σ, at which the value of the potential T is determined, ψ — angle between ρ and ρ'. Dividing (2.3) by ρ and integrating both parts of the equation over ρ from ρ to infinity, I. F. Monin [5] obtained

$$-T = \int_{\Sigma} \mu L d\Sigma, \text{ where } L = \frac{1}{\rho'} \ln \frac{\rho + \rho' - \rho\cos\psi}{\rho - \rho\cos\psi}. \qquad (2.4)$$

We obtain the following from (2.3)

$$\frac{\partial^2 T}{\partial \rho^2} = -\frac{1}{\rho^3}\int_{\Sigma}\frac{\mu}{r}\,d\Sigma + \frac{1}{\rho}\int_{\Sigma}\mu\,\frac{\partial}{\partial\rho}\left(\quad\right)d\Sigma \qquad (2.5)$$

After substituting(2.4) and (2.5) in the boundary condition (2.2) and changing to the surface Σ, utilizing the formula of Plemel, we obtain

$$\frac{2\pi\mu}{\rho}\cos\alpha - \frac{1}{\rho^3}\int_{\Sigma}\mu\left[6L - \frac{3}{2r} + \frac{\rho'^2 - \rho^2}{2r^3}\right]d\Sigma = \delta\rho, \qquad (2.6)$$

where α is the angle between ρ and the normal to the surface of the earth. Assuming that the error is several tens of Eötvös, we may write

$$\delta\rho = -\Delta g - \frac{\partial W}{\partial \rho}\zeta - \nabla\zeta.$$

The integral Equation (2.6) with respect to the density of the simple layer μ will be solved by the small parameter method [3]. For this purpose, we introduce the new auxiliary density χ of a simple layer determined by equation $\chi = \mu \sec\alpha$. Then the perturbing potential may be represented in a form

$$-T = \int_{\omega}\chi L\rho'^2 d\omega, \qquad (2.7)$$

where $d\omega$ is an elementary solid angle. Equation (2.6) may be written as follows:

$$\frac{2\pi\chi\cos^2\alpha}{\rho} - \frac{1}{\rho^3}\int_{\omega}\chi F\rho'^2 d\omega = \delta\rho, \qquad (2.8)$$

where $F = 6L - \frac{3}{2r} + \frac{\rho'^2 - \rho^2}{2r^3}$. To solve the integral equation (2.8), we assume another boundary surface Σ, for which the radius vector $\bar{\rho} = R + kH$, where $0 \leq k \leq 1$. We shall assume that the anomalies Δg are given, and we obtain a certain different auxiliary potential T. According to (2.7)

$$-T = \int_{\omega}\chi L\rho' d\omega. \qquad (*)$$

/ 31

where χ is the new density of the single layer which must satisfy the integral equation, similar to (2.8), that is equation

$$\frac{2\pi\bar{\chi}\cos^2\bar{\alpha}}{\rho} - \frac{1}{\rho^2}\int \chi F(\rho')\,d\omega = \delta p, \qquad (2.9)$$

$$F = 6\bar{L} - \frac{1}{2r}\left(1 + \frac{\bar{\rho}'^2 - \bar{\rho}^2}{2r^2}\right).$$

Since $\operatorname{tg}\alpha = k\operatorname{tg}\alpha$ [3], when $\alpha < 45°$ for all k within the limits established previously $\cos^2\bar{\alpha} = 1 - k^2\operatorname{tg}^2\alpha + k^4\operatorname{tg}^4\alpha - \ldots$

$$\bar{r} = r_0\left[1 + \frac{k}{2}\left(\frac{H' + H}{R}\right) + \frac{k^2}{2}\varkappa^2 - \frac{1}{4}k^3\varkappa^2\left(\frac{H' + H}{R}\right) - \frac{k^4\varkappa^4}{8} + \ldots\right].$$

$\varkappa = \frac{H' - H}{r_0}$. H' и H are the normal altitudes of running and fixed points, r_0 — distance between the points on the basic sphere corresponding to the running and fixed points, Σ. The perturbing potential $\bar{T}$ and the density $\bar{\chi}$ may be represented by the series

$$\bar{T} = \sum_0^\infty k^n T_n; \quad \bar{\chi} = \sum_0^\infty k^n \chi_n, \qquad (2.10)$$

where χ_n, T_n are the functions which do not depend on k and which are still unknown to us. Finding T_n, χ_n, we readily obtain $\bar{T}$, $\bar{\chi}$, since where $k = 1$ $\bar{\Sigma}$ changes into Σ and T, χ change into $\bar{T}$, $\bar{\chi}$, respectively. The integral equation (2.9) thus changes into (2.8) The perturbing potential T exists, just like the density χ, i.e., the series (2.10) converge when k = 1. When $0 < k < 1$ they converge absolutely, and they can be treated as finite sums. Representing $\bar{L}$ in the form of $\sum_0^\infty k^n L_n$, and substituting (2.10) in (*), we obtain

$$-\sum_0^\infty k^n T_n = \int_\sigma \left(\sum_0^\infty k^n \chi_n\right)\left(\sum_0^\infty k^n L_n\right)(R + kH')^2\,d\omega. \qquad (2.11)$$

In (2.11), equating the coefficients for equal powers of k, we obtain the following system of equations

$$-T_0 = R^2 \int_\omega \chi_0 L_0 \, d\omega;$$

$$-T_n = R^2 \int_\omega (\chi_0 L_n + \chi_1 L_{n-1} + \cdots + \chi_n L_0) \, d\omega, \qquad n = 1 \qquad (2.12)$$

In these expressions, beginning with T_1, small terms on the order of $T_0\left(\frac{H'}{R}\right)$ and higher are omitted.

As follows from (2.12) to determine the values of T_n, it is necessary to find the value of χ_n. This may be obtained by solving the integral equations, which we are now attempting to do. / 32

Equation (2.9) may be written in the form

$$\frac{2\pi}{R}(1 - k^2 \operatorname{tg}^2\alpha + k^4 \operatorname{tg}^4\alpha + \cdots) \times$$

$$\times \left(\sum k^n \chi_n\right)\left|1 - k\left(\frac{H}{R}\right) + k^2\left(\frac{H}{R}\right)^2 - \cdots\right| =$$

$$= \left|1 - 2k\left(\frac{H}{R}\right) + 3k^2\left(\frac{H}{R}\right)^2 - \cdots\right| \int_\omega \left(\sum_0^\infty \chi_n k^n\right) \times$$

$$\times \left(\sum k^n F_n\right)\left|1 - 2k\left(\frac{H'}{R}\right) + k^2\left(\frac{H'}{R}\right)^2\right| d\omega + \delta p.$$

Equating the coefficients for equal powers of k, we obtain a series of integral equations of the second kind with one and the same kernel F_1 and small terms on the order of $\chi_0\left(\frac{H}{R}\right)^2$ and higher are omitted. We thus have

$$\frac{2\pi}{R}\chi_n - \int_\omega \chi_n F_0 \, d\omega = \delta p_n, \qquad n(0, 1, 2 \ldots). \qquad (2.13)$$

where $\delta p_0 = \delta p_1 = \frac{2\pi}{R} \cdot \frac{H}{R}\chi_0 + \int_\omega \chi_0 F_1 \, d\omega - \frac{2}{R}\int_\omega (H - H')\chi_0 F_0 \, d\omega;$

$$\delta p_2 = \frac{2\pi}{R}\chi_0 \operatorname{tg}^2\alpha + \frac{2\pi}{R} \cdot \frac{H}{R}\chi_1 + \int_\omega (\chi_1 F_1 + \chi_0 F_2) \, d\omega -$$

$$- \frac{2}{R}\int_\omega (H - H')(\chi_1 F_0 + \chi_0 F_1) \, d\omega \qquad \text{etc.}$$

We shall search for the solution of the integral Equation (2.13) in the form $\chi_n = \sum_m (\chi_n)_m$, where $(\chi_n)_m$ is a spherical function of the mth order. As a result we obtain

$$\chi_n = (\chi_n)_1 - \frac{R}{64\pi^2} \int_\omega \delta\rho_n d\omega + \frac{R}{16\pi^2} \int_\omega \Phi(\psi)\, \delta\rho_n d\omega,$$

where $\sum_{m=2}^{\infty} \frac{(m+1)(2m+1)^2}{(m-1)(m+4)} P_m(\cos\psi)$, and $P_m(\cos\psi)$ is the Legendre polynomial of power m; $(\chi_n)_1$ is still arbitrary. In closed form $\Phi(\psi)$ can be written as follows

$$\Phi(\psi) = \frac{397}{20} - \frac{237}{25}\cos\psi - \frac{134}{5}\sin\frac{\psi}{2} - \frac{147}{2}\cos^2\psi +$$

$$+ 49\sin\frac{\psi}{2}\cos\psi + 147\sin\frac{\psi}{2}\cos^2\psi - \frac{3}{\sin\frac{\psi}{2}} -$$

$$- \frac{18}{5}\cos\psi \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) - \frac{147}{10}(3\cos\psi - 5\cos^3\psi)\ln\left(1 + \frac{1}{\sin\frac{\psi}{2}}\right)$$

$$L_0 = \frac{1}{R}\ln\left(1 + \frac{2R}{r_0}\right); \quad F_0 = 6L_0 - \frac{3}{2r_0};$$

$$L_1 = \frac{\varkappa}{R}; \quad F_1 = 6L_1 + \frac{3}{4r_0}\left(\frac{H'+H}{R}\right) + \frac{R\varkappa}{r_0^2};$$

$$I_2 = \frac{\varkappa^2 r_0}{4R} \cdot \frac{H\varkappa}{\quad} \int \quad \frac{\varkappa}{r} \quad \frac{\varkappa}{\quad} \quad H_0;$$

/33

$$I_3 \quad \rho \quad I \quad \quad \quad

3. Calculation of the Correction Factor in the Basic Boundary Condition

To calculate the small term in (2.1)

$$\left(\frac{\partial T}{\partial \rho}\right)_{\rho=R} + \frac{2T}{R} - \frac{\partial^2 T}{\partial \varphi^2}$$

it is necessary to know the perturbing potential and its derivatives in approximate form. If we assume the main term T_0 in its expansion is the approximate value of the potential, then to calculate T_0 it is not necessary to solve the integral equations, since the expression T_0 in terms of Δp was already found previously [1]. Solving the problem on determining the shape of the geoid, we obtain the boundary condition on the sphere for determining the perturbing potential. If it is not necessary that the masses of the geoid and the level ellipsoid be equal and if we assume that the centers of the masses coincide, we obtain

$$T_0 = \frac{1}{4\pi}\int_\sigma \Delta p \left[\Omega(\psi) - \frac{1}{4}\right] d\sigma - \frac{3}{2}(U_0 - W_0). \qquad (3.0)$$

Outside of a sphere of a radius R

$$T_0(\rho) = \frac{1}{4\pi}\int_\sigma \Delta p \left[\Omega(\rho, \psi) - \frac{R}{4\rho}\right] d\sigma - \frac{3}{2}(U_0 - W_0)\frac{R}{\rho}.$$

We thus have the following at points of the basic sphere

$$\frac{\partial T_0}{\partial \rho} = \frac{1}{4\pi}\int_\sigma \Delta p \left[\Gamma(\psi) - \frac{1}{4R}\right] d\sigma + \frac{3}{2}\left(\frac{U_0 - W_0}{R}\right), \qquad (3.1)$$

where $\Gamma(\psi) = \left[\frac{\partial \Omega(\rho, \psi)}{\partial \rho}\right]_{\rho=R}$. Differentiating (3.1) with respect to m and q, after changing the order of differentiation, we obtain

$$\frac{\partial^2 T_0}{\partial m \partial p} = \frac{\partial\left(\frac{\partial T_0}{\partial p}\right)}{R\partial\varphi} = -\frac{1}{4\pi R}\int_\sigma \Delta p \frac{d\Gamma}{d\psi}\cos\bar{\alpha}\, d\sigma, \qquad (3.2)$$

$$\frac{\partial^2 T_0}{\partial q \partial p} = \frac{\partial\left(\frac{\partial T_0}{\partial p}\right)}{R\cos\varphi\, d\lambda} = -\frac{1}{4\pi R}\int_\sigma \Delta p \frac{d\Gamma}{d\psi}\sin\bar{\alpha}\, d\sigma. \qquad (3.3)$$

Here φ and λ are the altitude and longitude of points on the sphere σ, and $\bar{\alpha}$ is its azimuth. We obtain the following from (3.0)

$$\frac{\partial T_0}{\partial m} = -\frac{1}{4\pi R}\int_\sigma \Delta p \frac{d\Omega}{d\varphi}\cos\bar{\alpha}\,d\sigma, \tag{3.4}$$

$$\frac{\partial T_0}{\partial q} = -\frac{1}{4\pi R}\int_\sigma \Delta p \frac{d\Omega}{d\varphi}\sin\bar{\alpha}\,d\sigma. \tag{3.5}$$

Thus, everything necessary for calculating the correction coefficient in (2.1) is given by the formulas (3.2), (3.3), (3.4) and (3.5).

The results obtained in [1] make it possible to simplify somewhat the solution of the basic problem of this study. If we assume H = H' = 0, then the boundary condition (2.2) will be given at points on the surface of a sphere with the radius R, and we obtain only one integral equation $\frac{2\pi}{R}\chi_0 - \int \chi_0 F_0 d\omega$. /34

The solution of the boundary value problem will be $R^2 \int \chi_0$ according to (2.7) and we have already obtained the solution previously in the form $-\frac{1}{4\pi}\int_\sigma \delta p \left|\Omega(\psi) - \frac{1}{\ell}\right| d\sigma + X_1$. It is true that the harmonics of the zero and first orders were absent and here we retain them. In this way we do not need to assume that the masses of the earth and the ellipsoid are equal, or that the centers of masses of both bodies coincide. We thus have

$$R^2\int_\sigma \chi_0 L_0 d\omega = \frac{1}{4\pi}\int_\sigma \delta p\left|\Omega(\psi) - \frac{1}{\ell}\right| d\sigma + X_1. \tag{3.6}$$

Assuming $\delta p_2 \ldots$ in the boundary condition instead of $\delta p\ \delta p_1$, on the basis of the preceding considerations we obtain

$$R^2\int_\sigma \chi_1 L_0 d\omega = \frac{1}{4\pi}\int_\sigma \delta p_1\left|\Omega(\psi) - \frac{1}{\ell}\right| d\sigma + X_1 \quad \text{etc.} \tag{3.7}$$

Since $T = T_0 + T_1 + \ldots$, according to (2.12), (3.4), (3.7), we may find

$$T = \frac{1}{4\pi} \int_{\sigma} (\Delta p - \delta p_1 - \delta p_2 - \cdots) \left[\Omega(\psi) - \frac{1}{l}\right] d\sigma -$$

$$- \frac{3}{2} (U_0 - W_0) + Y_1 - \int_{\sigma} \delta p (L_1 + L_2 + \cdots) d\sigma - \tag{3.8}$$

$$- \int_{\sigma} \zeta_1 (L_1 + L_2 + \cdots) d\sigma - \int_{\sigma} \zeta_2 (L_1 + \cdots) d\sigma + \cdots$$

Y_1 is an arbitrary spherical function of the first order. The formula (3.8) solves the problem posed.

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